REGULAR NUMBER OF TOTAL BLOCK GRAPH OF A GRAPH

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ABSTRACT:

For any $(p\ ,q)$ graph G, the vertices and blocks of a graph are called its members. The total block graph $T_B(G)$ of a graph G as the graph whose points can be put in one-to-one correspondence with the set of points and blocks of G in such a way that two points of $T_B(G)$ are adjacent if and only if the corresponding members of G are adjacent or incident. The regular number of $T_B(G)$ is the minimum number of subsets into which the edge set of $T_B(G)$ should be partitioned so that the subgraph induced by each subset is regular and is denoted by $r_{T_B}(G)$. In this paper some results on $r_{T_B}(G)$ were obtained and expressed in terms of elements of G.

Key words : Regular number / Total block graph / binary tree / domination number/ total domination number / independent domination number / edge domination number / cototal domination number / connected domination number.

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Introduction :

In this paper, we follow the notations of [4]. All the graphs considered here are simple, finite, and non-trivial. As usual p and q denote the number of vertices and edges of a graph G respectively. The maximum degree of a vertex in G is denoted by $\Delta(G)$. A vertex v is called a cutvertex if removing it from G increases the number of components of G. A graph G is called trivial if it has no edges. The maximum distance between any two vertices in a G is called a diameter and is denoted by diam(G). The path and tree numbers were introduced by Stanton James and Cown in[18]. The independence number $\beta_1(G)$ is the maximum cardinality of an edge independent set in G. Let $G = (V, E)$ be a graph. A set $D' \subseteq V$ is said to be a dominating set of G, if every vertex in $(V - D')$ is adjacent to some vertex in D'. The minimum cardinality of vertices in such a set is called the domination number of G and is denoted by $\gamma(G)$. The dominating graph was studied by V.R.Kulli and K.M.Niranjan in[27]. A dominating set is said to be total dominating set of G, if N$(D') = V$ or equivalently, if for every $v \in V$, there exists a vertex $u \in D'$, $u \neq v$, such that u is adjacent to v. The total domination number of G, denoted by $\gamma_t(G)$ is the minimum cardinality of total dominating set of G. A set with minimum cardinality among all the maximal independent set of G is called minimum independent dominating set of G. The cardinality of a minimum independent dominating set is called independent domination number of the graph G and it is denoted by $i(G)$. On complementary graphs was studied by E. A. Nordhaus and J. W. Gaddum in[2]. The regular number of graph valued function was studied by M.H.Muddebihal, Abdul Gaffar, and Shabbir Ahmed in[11] and also developed in[12, 13, 14, 15]. Domination related parameters are now well studied in graph theory. The total domination $\gamma_t(G)$ was studied by M.H.Muddebihal, Srinivasa.G, and A.R.Sedamkar in[9]. Total domination in graphs was studied by E.J.Cockayne, R.M.Dawes, and S.T.Hedetniemi in[1]. This concept was studied by M.A.Henning in[6] and was studied, for example in[7, 8, 10, 16, 26]. A dominating set D of L(G) is a regular total dominating set (RTDS) if the induced subgraph $< D >$ has no isolated vertices and $\deg(v) = 1, \forall v \in D$. The regular total domination in line graphs was studied by M.H.Muddebihal, U.A.Panfarosh and Anil.R.Sedamkar in[10]. Total domination and total domination subdivision numbers of graphs were studied by O. Favaron, H. Karami and S. M. Sheikholeslami in[17]. On block-cutvertex trees was studied by V.R.Kulli in[22]. On line graphs with crossing number was studied by V.R. Kulli, D.G.Akka and L.W. Bieneke in[24] and was studied for example in[23, 25]. A dominating set D of G is a cototal dominating set if the induced subgraph $< V - D >$ has no isolated vertices. The cototal domination number $\gamma_{cot}(G)$ of G is the minimum cardinality of a cototal dominating set. A set $D \subseteq E$ is said to be a edge dominating set of G, if every edge in $(E - D)$ is adjacent to some edge in D. The minimum cardinality of a edge dominating set is called edge domination number of G and is denoted by $\gamma^{'}(G)$. The edge dominating graph was studied by S.Mitchell and S.T.Hedetniemi in[19] .

A dominating set D is said to be connected dominating set if the induced subgraph $< D >$ is connected. The minimum cardinality of a minimal connected dominating set is called a connected domination number and is denoted by $\gamma_C(G)$. The connected dominating graph was studied by E.Sampath Kumar and H.B Walikar in[3] and was studied for example in[5, 20, 21].

4. *Results :*

The following results are obvious, hence we omit its proof .

Theorem 1 : For any graph G, $r_{T_B}(G) = 1$.

if and only if $T_B(G)$ is regular.

The following theorem gives a clear equality of the regular number of total block graph of a path with $p \geq 3$ vertices.

Theorem 2 : For any path P_p with $p \geq 3$ then $r_{T_B}(P_p) = 3$.

Proof : Let $P_p : e_1 = v_1v_2 , e_2 = v_2v_3 , e_3 = v_3v_4 , \ldots , e_{p-2} = v_{p-2}v_{p-1} , e_{p-1} = v_{p-1}v_p$ be a path and every edge is a block. In $T_B(P_p)$, V[$T_B(P_p)$] = V(P_p) ∪{ $v_1^{'} , v_2^{'} , v_3^{'} , \ldots , v_{p-1}^{'}$}. Let $e_1^{'} = v_1^{'}v_2^{'} , e_2^{'} = v_2^{'}v_3^{'} , e_3^{'} = v_3^{'}v_4^{'} , \ldots , e_{p-3}^{'} = v_{p-3}^{'}v_{p-2}^{'} , e_{p-2}^{'} = v_{p-2}^{'}v_{p-1}^{'} , e_1^{''} = v_1v_1^{'} , e_2^{''} = v_2v_2^{'} , e_3^{''} = v_3v_3^{'} , \ldots , e_{p-3}^{''} = v_{p-3}v_{p-3}^{'} , e_{p-2}^{''} = v_{p-2}v_{p-2}^{'} , e_{p-1}^{''} = v_{p-1}v_{p-1}^{'}$; $e_1^{'''} = v_2v_1^{'} , e_2^{'''} = v_3v_2^{'} , e_3^{'''} = v_4v_3^{'} , \ldots , e_{p-3}^{'''} = v_{p-2}v_{p-3}^{'} , e_{p-2}^{'''} = v_{p-1}v_{p-2}^{'} , e_{p-1}^{'''} = v_pv_{p-1}^{'}$ and $e_1 = v_1v_2 , e_2 = v_2v_3 , e_3 = v_3v_4 , \ldots , e_{p-2} = v_{p-2}v_{p-1} , e_{p-1} = v_{p-1}v_p$ are the edges of $T_B(P_p)$. Let $F_1 = \{ v_1e_1v_2e_2v_3\, e_3v_4e_4v_5 \ldots v_{p-3}e_{p-3}v_{p-2}e_{p-2}v_{p-1}e_{p-1}v_p e_{p-1}^{'''}v_{p-1}^{'}e_{p-2}^{'}v_{p-2}^{'}e_{p-3}^{'}v_{p-3}^{'} \ldots v_5^{'}e_4^{'}v_4^{'}e_3^{'}v_3^{'}e_2^{'}v_2^{'}e_1^{'}v_1^{'}e_1^{''}v_1 \}$ is a 2-regular block.

$F_2 = \{ v_1e_1^{''}v_1^{'} , v_2e_2^{''}v_2^{'} , v_3e_3^{''}v_3^{'} , \ldots , v_{p-2}e_{p-2}^{''}v_{p-2}^{'} , v_{p-1}e_{p-1}^{''}v_{p-1}^{'} \}$ and

$F_3 = \{ v_2 e_1^{'''} v_1^{'} , v_3 e_2^{'''} v_2^{'} , v_4 e_3^{'''} v_3^{'} , \ldots , v_{p-1} e_{p-2}^{'''} v_{p-2}^{'} , v_p e_{p-1}^{'''} v_{p-1}^{'} \}.$

Let F be the minimum regular partition of $T_B(P_p)$.

Thus

$$F = \{ F_1 , F_2 , F_3 \}.$$

$$r_{T_B}(P_p) = |F| = |F_1 , F_2 , F_3|.$$

Hence,

$$r_{T_B}(P_p) = 3.$$

We establish the following theorem to prove our further theorem.

Theorem 3 : For any non-trivial tree T with n-cutvertices with same degree then,

$$r_{T_B}(T) \leq q - \beta_1(T) + 2.$$

Proof : Let S be a maximum edge independent set in T. Then $E - S$ has at most $|E - S|$ edge independent sets.

Thus

$$r_{T_B}(T) \leq |E - S| + 2.$$

$$r_{T_B}(T) \leq q - \beta_1(T) + 2.$$

Now the following result determines the upper bound on $r_{T_B}(T)$.

Theorem 4: For any non-trivial tree T then

$$r_{T_B}(T) \leq 2q - p + 2.$$

Proof : By Theorem 3, we have

$$r_{T_B}(T) \leq q - \beta_1(T) + 2.$$

Since, $\beta_1(T) \geq \gamma^{'}(T)$.

Where $\gamma^{'}(G)$ is the edge domination number of G.

This implies,

$$r_{T_B}(T) \leq q - \gamma^{'}(T) + 2.$$

Also $p - q \leq \gamma^{'}(T)$.

Thus,

$$r_{T_B}(T) \leq q - (p - q) + 2.$$

$$r_{T_B}(T) \leq q - p + q + 2.$$

$$r_{T_B}(T) \leq 2q - p + 2.$$

In the next result we obtain Nordhaus-Gaddum type result on $r_{T_B}(T)$.

Theorem 5 : For any non-trivial tree T then

$$r_{T_B}(T) + r_{T_B}(\bar{T}) \le p(p-3) + 4.$$

Proof : By Theorem 4, we have

$$r_{T_B}(T) \le 2q - p + 2.$$

$$r_{T_B}(\bar{T}) \le 2\bar{q} - p + 2.$$

$$r_{T_B}(T) + r_{T_B}(\bar{T}) \le 2(q + \bar{q}) - 2p + 4.$$

$$r_{T_B}(T) + r_{T_B}(\bar{T}) \le 2\binom{p}{2} - 2p + 4.$$

$$r_{T_B}(T) + r_{T_B}(\bar{T}) \le 2\frac{p(p-1)}{2} - 2p + 4.$$

$$r_{T_B}(T) + r_{T_B}(\bar{T}) \le p(p-1) - 2p + 4.$$

$$r_{T_B}(T) + r_{T_B}(\bar{T}) \le p(p-1-2) + 4.$$

$$r_{T_B}(T) + r_{T_B}(\bar{T}) \le p(p-3) + 4.$$

Now we proceed with the regular number of total block graph of a binary tree.

Theorem 6 : For any non-trivial binary (p, q) tree T then

$$r_{T_B}(T) = 3 \;\; ; \text{ if } p = 3.$$

$$= 5 \;\; ; \text{ if } p \ge 5.$$

Proof : Let T be a non trivial binary tree. Then, we consider the following cases.

Case 1: If $p = 3$, let v_1, v_2, v_3 be the vertices of T, such that $\deg(v_1) = \deg(v_3) = 1$ and $\deg(v_2) = 2$. Let $e_1 = v_1v_2$ and $e_2 = v_2v_3$ be the edges of T. Since every edge is a block in T. Now in $T_B(T)$, $V[T_B(T)] = \{ v_1, v_2, v_3 \} \cup \{ v_1', v_2' \}$ where v_1', v_2' be the corresponding vertices of the edges e_1 and e_2 respectively. Further, e_1 and e_2 forms the blocks with the vertices v_1' and v_2'. Let $F_1 = \{ v_1v_1'v_2 \}$, $F_2 = \{ v_2v_2'v_3 \}$ and $F_3 = \{ v_1'v_2' \}$. Let F be the minimum regular partition of $T_B(T)$.

Thus

$$r_{T_B}(T) = |F|$$

$$= 3.$$

Case 2 : If $p = 5$, let v_1, v_2, v_3, v_4, v_5 be the vertices of T, such that $\deg(v_1) = 2$, $\deg(v_2) = 3$ and $\deg(v_3) = \deg(v_4) = \deg(v_5) = 1$. Let $e_1 = v_1v_2$, $e_2 = v_1v_3$, $e_3 = v_2v_4$ and $e_4 = v_2v_5$ be the edges of T. Now, in $T_B(T)$, $V[T_B(T)] = \{ v_1, v_2, v_3, v_4, v_5 \} \cup \{$

v_1', v_2', v_3', v_4' }, where v_1', v_2', v_3', v_4' be the corresponding vertices of the edges e_1, e_2, e_3 and e_4 respectively.

Let $F_1 = \{ v_1 v_1' v_3 , v_4 v_4' v_2 \}$, $F_2 = \{ v_1 v_2 v_2' \}$, $F_3 = \{ v_2 v_3' v_5 \}$, $F_4 = \{ v_4' v_3' v_2' \}$, $F_5 = \{ v_1' v_2' \}$.

Let F be the minimum regular partition of $T_B(T)$.

Thus

$$r_{T_B}(T) = |F|.$$
$$= 5.$$

If $p = 7$, let $v_1, v_2, v_3, v_4, v_5, v_6, v_7$ be the vertices of T, such that $\deg(v_1) = 2$, $\deg(v_2) = \deg(v_3) = 3$ and $\deg(v_4) = \deg(v_5) = \deg(v_6) = \deg(v_7) = 1$. Let $e_1 = v_1 v_2$, $e_2 = v_1 v_3$, $e_3 = v_2 v_4$, $e_4 = v_2 v_5$, $e_5 = v_3 v_6$, and $e_6 = v_3 v_7$ be the edges of T. Now, in $T_B(T)$, $V[T_B(T)] = \{ v_1, v_2, v_3, v_4, v_5, v_6, v_7 \} \cup \{ v_1', v_2', v_3', v_4', v_5', v_6' \}$, where $v_1', v_2', v_3', v_4', v_5', v_6'$ be the corresponding vertices of the edges $e_1, e_2, e_3, e_4, e_5, e_6$ respectively.

Let $F_1 = \{ v_1 v_2 v_1' , v_3 v_7 v_6' \}$, $F_2 = \{ v_2 v_5 v_4' , v_3 v_6 v_5' \}$, $F_3 = \{ v_2 v_4 v_3' , v_1 v_3 v_2' \}$,
$F_4 = \{ v_1' v_3' v_4' , v_2' v_5' v_6' \}$ and $F_5 = \{ v_1' v_2' \}$.

Let F be the minimum regular partition of $T_B(T)$.

Hence

$$r_{T_B}(T) = |F|.$$
$$= 5.$$

If $p > 7$. In T, $\Delta(T) = 3$, and there exists only one vertex of degree 2, let v be a vertex with $\deg(v) = 2$. $\mathrm{N}(v) = v_1', v_2'$ and in $T_B(T)$ which forms a block of 1-regular. Clearly, those vertices of degree 3 which forms the blocks of 2-regular which are adjacent to each other. Hence these adjacent blocks belongs to either F_1, F_2, F_3 or F_4. Hence in general for ≥ 5, we have

$$r_{T_B}(T) = |\{ F_1, F_2, F_3, F_4 \}| + 1.$$
$$= 4 + 1.$$
$$= 5.$$

In the above all cases, we have the required result.

Now we give the exact value of $r_{T_B}(T)$, with n-cutvertices and degree of each cutvertex is 3.

Theorem 7 : For any non-trivial tree T, with n-cutvertices and degree of each cutvertex equals to 3, then

$r_{T_B}(T) = 4.$

Proof : For any tree T, $V_1 = \{ v_1 , v_2 , v_3 , ... , v_n \} \subseteq V(T)$ be the set of all cutvertices. Suppose, $\deg(v_1) = \deg(v_2) = \deg(v_3) = , . . . , = \deg(v_n) = 3$. Then, in $T_B(T)$, $\forall\ v_i \in V_1$, $1 \leq i \leq n$, gives the blocks which are 2-regular. Further, in $T_B(T)$, since each cutvertex of $T_B(T)$ is incident with 3 blocks which are 2-regular and every block is adjacent to each other, hence these adjacent edges also forms a 2-regular. Let F be the minimum regular partition of $T_B(T)$. Since every block is 2-regular, the subgraph induced by each subset is 4 times of K_3 . Therefore, each of the blocks B_1 , B_2 , B_3 and B_4 belongs to different sets F_1 , F_2 , F_3 and F_4 of F respectively.

Hence

$$r_{T_B}(T) = |F|.$$
$$= 4.$$

Now we give the sharp value of regular number of total block graph of a tree with n-cutvertices with same degree and degree of each cutvertex is more than 3.

Theorem 8 : For any non-trivial tree T with n-cutvertices with same degree and degree of each cutvertex is more than 3, then $r_{T_B}(T) =$ degree of cutvertex + 2.

Proof : For any tree T, $V_1 = \{ v_1 , v_2 , v_3 , ... , v_n \}$ be the subset of $V(T)$ be the set of all cutvertices. Suppose, $\deg(v_1) = \deg(v_2) = \deg(v_3) = , . . . , = \deg(v_n) = m$. Then, in $T_B(T)$, $\forall$ $v_i \in V_1$ such that $1 \leq i \leq n$, gives m-number of 2-regular blocks which are adjacent to each other. Further, in $T_B(T)$, on every cutvertex there are m-number of 2-regular blocks. And the m-number of 2-regular blocks B_1 , B_2 , B_3 , . . . , B_m belongs to different sets F_1 , F_2 , F_3 , . . . , F_m respectively. And these blocks are adjacent to each other and hence these adjacent edges which forms a complete graph and degree of each vertex is of $(m-1)$, and these complete graphs are also adjacent to each other. Hence, these complete graphs belongs to either F_{m+1} or F_{m+2} .

Hence

$$r_{T_B}(T) = |\{ F_1 , F_2 , F_3 , . . . , F_m , F_{m+1} , F_{m+2} \}|.$$
$$r_{T_B}(T) = |\{ F_1 , F_2 , F_3 , . . . , F_m \}| + 2.$$
$$r_{T_B}(T) = m + 2.$$

$r_{T_B}(T) =$ degree of cutvertex $+ 2$.

In the following theorem we establish the relationship between $r_{T_B}(T)$ and $\gamma(T)$.

Theorem 9 : For any non-trivial tree T with n-distinct cutvertices then, $r_{T_B}(T) > \gamma(T)$.

Proof : For any tree T, let $D = \{ v_1, v_2, v_3, \ldots, v_n \}$ be the set of non-end vertices which are cutvertices in T. Suppose every vertex of D is adjacent to atleast one end vertex. Then D is a γ-set of T. Suppose, $\deg(v_1) < \deg(v_2) < \deg(v_3) < \ldots < \deg(v_n)$. Now, $\forall\, v_i \in D$ such that $1 \le i \le n$ and $\deg(v_i) \ge 2$. Let $\deg(v_n) = \Delta(T) = n$ in T. Then in $T_B(T)$, the regular partitions on this vertex v_n has n-number of 2-regular blocks with one more $(n-1)$ regular graphs. Then clearly in $T_B(T)$ have more than n-number of regular partitions.

Hence

$$r_{T_B}(T) > \gamma(T).$$

In the following theorem we developed the above result of Theorem 9 in terms of equality.

Theorem 10 : For any non-trivial tree T, $T \ne P_p$ with $p \ge 10$ then $r_{T_B}(T) \ge \gamma(T)$.

In the next result we developed a relationship between $r_{T_B}(T)$ and $\gamma_t(T)$.

Theorem 11 : For any non-trivial tree T, with n-distinct degrees cutvertices, then

$$r_{T_B}(T) \ge \gamma_t(T).$$

Proof : Suppose $T = P_p$,

By Theorem 2, we have

$$r_{T_B}(P_p) = 3.$$

If $= 6$, then $\gamma_t(P_6) = 4$ and

$$r_{T_B}(P_6) = 3. \text{ Hence, } r_{T_B}(P_6) < \gamma_t(T).$$

Which is a contradiction to our hypothesis.

Suppose T be any non-trivial tree with $p = 2$, then $\gamma_t(T) = 1$ and $r_{T_B}(T) = 1$. Let $p = 3, 4, 5,$ then

By Theorem 2, we have

$r_{T_B}(T) = 3$ and $\gamma_t(T) \le 3$.

Hence

$$r_{T_B}(T) \ge \gamma_t(T).$$

Let $D = \{ v_1, v_2, v_3, . . . , v_n \} \subseteq V(T)$ be the minimal dominating set of T. Suppose $< D >$ has no isolates. Then D is a γ_t-set. Suppose, $\{ v_1, v_2 \} \in D$ which are the distinct cutvertices of T, such that $\deg(v_1) = 2$ and $\deg(v_2) = 3$, then clearly $\gamma_t(T) = 2$ and $r_{T_B}(T) = 5$.

Thus

$$r_{T_B}(T) \geq \gamma_t(T).$$

In succession, suppose $\{ v_1, v_2, v_3 \} \in D$ and v_i belongs to cutvertices such that $1 \leq i \leq 3$, $\forall v_i \in V(T)$ and $\deg(v_1) = 2$, $\deg(v_2) = 3$and $\deg(v_3) = 4$, then $\gamma_t(T) = 3$ and $r_{T_B}(T) = 6$.

Hence

$$r_{T_B}(T) \geq \gamma_t(T).$$

Let T has $\{ v_1, v_2, v_3, . . . , v_n \}$ cutvertices then clearly $\gamma_t(T) = n$ and $\deg(v_1) = 2$, $\deg(v_2) = 3$ $\deg(v_3) = 4, . . . , \deg(v_n) = n + 1$ and the vertex v_n has $(n + 1)$ regular partitions along with another partition which is a complete graph of n-regular and each partition of $(n + 1)$ is K_3 .

It follows that,

$$r_{T_B}(T) \geq (n + 1) + 1.$$

$$r_{T_B}(T) \geq n + 2.$$

This implies,

$$r_{T_B}(T) \geq n.$$

Hence,

$$r_{T_B}(T) \geq \gamma_t(T).$$

The following theorem gives a relationship between $r_{T_B}(T)$ and $\gamma_C(T)$.

Theorem 12 : For any non-trivial tree T with n-distinct cutvertices then,

$$r_{T_B}(T) \geq \gamma_C(T).$$

Proof : Let $V_1 = \{ v_1, v_2, v_3, . . . , v_n \} \subseteq V(G)$ be the set of all non-end vertices in G. Suppose there exists a minimal set of vertices S = $\{ v_1, v_2, v_3, . . . , v_j \} \subseteq V_1$ such that N$[v_i]$= $V(G)$ $\forall v_i \in$ S , $1 \leq i \leq j$. Then S forms a minimal dominating set of G.

Further if the subgraph $< S >$ has exactly one component, then S itself is a connected dominating set of G. Suppose S has more than one component. Then attach the minimum set of vertices S' from $V - S_1$ to S which are in every u-v path of G, $\forall u, v \in S'$ gives a single component $S_1 = S \cup S'$. Clearly S_1 forms a γ_c-set of G. For a regular number of a non-trivial tree, we consider the following cases.

Case 1. Suppose a non-trivial tree T is a path. Then $\gamma_c(P_p) = P - 2$ and by Theorem 2, $r_{T_B}(G) = 3$, which gives $\gamma_C(P) > r_{T_B}(P)$. Hence $T \neq P_p$, $p \geq 6$ vertices.

Case 2. Suppose a non-trivial tree T is not a path .Then there exists at least one vertex v of degree ≥ 3, such that $\deg(v) = \Delta(T)$, $\forall\, v \in V_1$, then $\Delta(T) < |V_1|$. By Theorem 8, $r_{T_B}(T) < \gamma_C(T)$. Thus from the above two cases, we have $\Delta(T) > |n|$. Then by Theorem 8, $r_{T_B}(T) = \Delta(T) + 2 > |n|$ which gives $r_{T_B}(T) \geq \gamma_C(T)$.

In the next result we discuss a relationship between $r_{T_B}(T)$ and $i(T)$ where i is independent domination number.

Theorem 13 : For any non-trivial tree T, with $T \neq P_p$ with $P > 6$ vertices $r_{T_B}(T) \geq i(T)$.

Proof **:** Suppose for any non trivial tree $T = P_p$ with $p = 2, 3, 4, 5$. Then by Theorem 2, $r_{T_B}(P_p) = 3$ and $i(P_p) = 1$ or 2. If T is a path with $p \geq 6$ vertices, then by Theorem 2, $r_{T_B}(T)$ has a fixed positive integer 3 where as independence domination number of P_p increases. Hence $r_{T_B}(P_p) \leq i(P_p)$. Hence $T \neq P_p$ with $p > 6$ vertices. Further for any tree which is not a path with $p < 6$ vertices. Let $V_1 = \{ v_1, v_2, v_3, \ldots, v_n\} \subseteq V(T)$ be the set of all non end vertices which are cutvertices.

Let $V_2 = V(T) - V_1$be the set of end vertices. Suppose $V_1' \subseteq V_1$ and N(V_1') $= V(T)$. Then V_1' is a minimal dominating set of T. If $< V_1' >$is totally disconnected, then V_1' is an independent domination set of T.

Suppose $V(T) = \{ v_1, v_2, \ldots, v_m\}$ and $E(T) = \{e_1, e_2, \ldots, e_{m-1}\}$. Then in $T_B(T)$, $e_i \leftrightarrow v_i'$ where $\forall\, v_i' \in V[T_B(T)]$ such that $V[T_B(T)] = V(T) \cup E(T)$ and $\forall\, e_i \in E(T)$. Let $\{v_j\}$ be the set of vertices with maximum degree. In $T_B(T)$, $\{v_j'\}$ forms a maximal complete graph $K_{v_j'}$ as an induced subgraph of $T_B(T)$. Further let $\{v_k''\} \in N(v_j)$ and $\{v_j\} \cup \{v_j'\} \cup \{v_k''\}$ form a set of edge disjoint complete induced subgraph as K_3. Since v_j', $v_k'' \in N(v_j)$ and $\{v_j'\} \cap \{v_k''\} = v_j$, then each K_3 belongs to different regular partition.

Thus $\{v_1 \cup v_1^{'} \cup v_k\}$,$\{ v_2 \cup v_2^{'} \cup v_2^{''} \}$, . . . , $\{v_j \cup v_j^{'} \cup v_k^{'}\}$ are the number of edge disjoint K_3 and belongs to F_1 , F_2 , . . . , F_n such that $F = \{v_j^{'}\}$, F_1 , F_2 , . . . , F_n . Hence $|F| = |v_j^{'}, F_1, \ldots, F_n| < |V_1^{'}|$ gives $r_{T_B}(T) > i(T)$.

Now we establish a relationship between $r_{T_B}(T)$ and $\gamma^{'}(T)$.

Theorem 14 : For any non-trivial tree T with $T \neq P_p$ with $p \geq 11$,

$$r_{T_B}(T) \geq \gamma^{'}(T).$$

Proof : Suppose $T = P_p$ with $1 \leq p \leq 10$.

Then by Theorem 2, $r_{T_B}(P_p) \geq \gamma^{'}(T)$.

Now assume $T = P_p$ with $p \geq 11$ vertices.

Then by Theorem 2, $r_{T_B}(P_p) = 3$ and $\gamma^{'}(T) \geq 4$.

Hence it contradict the fact of the theorem . Suppose a non trivial tree is not a path and there exists at least one vertex $v \in \Delta(T)$. Then by Theorem 8, $r_{T_B}(T) = \Delta(T) + 2$.

Since $\gamma^{'}(T) < \Delta(T)$, then

$$r_{T_B}(T) \geq \gamma^{'}(T).$$

Finally we discuss the relationship between $r_{T_B}(T)$ and $\gamma_{cot}(T)$.

Theorem 15 : For any non-trivial (p, q) tree T with $p \geq 4$ vertices, $r_{T_B}(T) \leq \gamma_{cot}(T)$.

Proof : Suppose $p = 2, \gamma_{cot}(T)$ doesnot exists, for $p = 3$. Then $r_{T_B}(T) > \gamma_{cot}(T)$. Hence a non-trivial tree has $p \geq 4$ vertices. Let $V_1 = \{ v_1 , v_2 , \ldots , v_n\}$ be the set of all end vertices of a tree T.

Suppose $V_2 = V(T) - V_1$ and consider $V_2^{'} \subseteq V_2$ be the set of vertices which are adjacent to end vertices . Again we consider $V_2^{''} \subseteq V_2$ such that $\forall\, v_i \in V_{2j}^{''}$, $\forall\, v_j \in V_2^{'}$, $d(v_j, v_i) \geq 2$.

Since $< V(T) - \{V_1 \cup V_2^{''}\} >$ has no isolates, then $V_1 \cup V_2^{''}$ form a cototal dominating set of tree T.

Suppose $V_2 = \{ v_1 , v_2 , \ldots , v_k\}$ and $\deg(v_1) \geq \deg(v_2) \geq \ldots \geq \deg(v_k)$. Then there exists $V_3 = \{ v_1 , v_2 , \ldots , v_m\} \subseteq V_2$ such that $\deg(v_m) = \Delta(T)\ \forall\, v_m \in V_3$. In $T_B(T)$, $V[T_B(T)] = V(T) \cup E(T)$ such that $\{v_1^{'}, v_2^{'}, \ldots, v_q^{'}\}$ be the set of vertices in $T_B(T)$ which preserve one-to-one correspondence with the edge set of T. Clearly $V[T_B(T)] = V_1 \cup V_2 \cup \{ v_1^{'}, v_2^{'}, \ldots, v_q^{'}\}$. If for some $v_n \in V_1$ which are adjacent to some $v_m \in V_3$, then $\{v_n\}$ form a edge disjoint complete graph $K_{\Delta(T)-1}$ regular.

Since $\{v_n\} \cup \{v_m\} \cup \{v_1^{'}, v_2^{'}, \ldots, v_i^{'}\}$, $1 \le i \le q$ form 2-regular blocks which are adjacent to each other. Then these blocks belongs to different sets $F_1, F_2, \ldots, F_m$ and $K_{\Delta(T)-1}$ belongs F_{m+1} sets. Hence $|V_1 \cup V_2^{''}| \ge |F_1, F_2, \ldots, F_m| \cup |K_{\Delta(T)-1}|$, which gives $r_{T_B}(T) \ge \gamma_{cot}(T)$.

5.Conclusion:

We studied the property of our concept by applying to some standard graphs. We also establish the regular number of total block graph of some standard graphs. Further we develop the upper bound in terms of minimum edge independence number of G and vertices of G. We establish some properties of this graph. Also many results are sharp.

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